Stabilizability of Sandpiles

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Contents

Abstract

The main aim of this project is to prove a non-trivial upper bound for the critical density when stabilizing i.i.d. sandpiles on $\mathbb{Z}^2$, as proved by Hough, Jerison, and Levine in [1]. More detail is provided to make the content accessible at an undergraduate level and familiarize the reader with the model itself as well as pertaining results.

To this end, Section 1 establishes the model and introduces an elementary necessary condition for stabilizability. The upper bound itself is then introduced in Section 2 to motivate the reader as we begin chararcterizing $\mathscr{H}^1(\mathbb{Z}^2)$ (the space comprising of harmonic modulo one functions in ℓ^1 on the square lattice). The utility of the aforementioned characterization then becomes apparent in Section 3 as we introduce results for such functions which are paramount in the upper bound proof. Section 3 is finished by the completion of the theorem in question, using an arbitrarily chosen function of $\mathscr{H}^1(\mathbb{Z}^2)$ for completeness.

Finally, Section 4 proves an unseen analogous result on $\mathbb{Z} \times \mathbb{Z}_k$, employing both similair and unique techniques to do so.

1 Introduction

The abelian sandpile model was first introduced in a paper by Bak, Tang, and Wiesenfeld in 1987 to display self-organized criticality [6]. This property of dynamical systems is considered to be a mechanism by which complexity arises in nature. Intuitively, the model describes grains of sand placed on a grid, where each site has an associated value corresponding to the number of particles on the pile. Once a certain threshold is reached, the site topples, distributing particles to its neighbours.

Throughout the paper, sandpiles are formally defined only in the case of the square lattice, so in the interest of brevity and readability it may be assumed σ is a sandpile on $\mathbb{Z}^2$ unless stated otherwise.

1.1 Sandpiles on the Square Lattice

Definition 1.1. *A sandpile on the square lattice is a function $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$, such that $\sigma(x)$ represents the number of sand grains at the site x. The sandpile is said to be* ***stable*** *if $\sigma(x) \leq 3$ for all sites x, otherwise it is said to be* ***unstable****.*

At sites x such that $\sigma(x) \geq 4$, we may perform a ***toppling*** *which distributes a grain of sand to each adjacent site. Toppling at site x corresponds to a change in the sandpile σ as follows:*

$$\sigma(x) \to \sigma'(x) = \sigma(x) - 4$$
$$\sigma(y) \to \sigma'(y) = \sigma(y) + 1$$

for each neighbour y of x. Notice that a sandpile σ is stable if and only if there are no legal topplings. A sandpile σ is an ***i.i.d. sandpile*** *if the number of particles at each site are independent and identically distributed random variables.*

Iterative Procedure *For a sandpile σ, let $\sigma_0 = \sigma$ and then iteratively define σ_{n+1} by simultaneously toppling each site of σ_n with at least 4 grains of sand. This process defines a sequence of sandpiles $(\sigma_n)_{n\in\mathbb{N}}$.*

We say that σ ***stabilizes*** *(or, equivalently, is stabilizable) if it is possible to obtain a stable configuration from σ by toppling each vertex finitely many times. Refer to the resulting stable configuration as the* ***stabilization*** *of σ, denoted by σ_∞.*

The following example illuminates the fact that a sandpile which stabilizes may have a non-finite number of iterations. In other words, *non-stabilizability* is a strictly stronger property than 'taking infintely long to stabilize'. As an example, we now provide an initial configuration which doesn't stabilize in finite time, however is still stabilizable.

Example 1.1. *The configuration σ defined by*

$$\sigma(i,j) = \begin{cases} 4 & \text{if } i = j = 0 \\ 3 & \text{if } i = 0, j \neq 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

is such that σ_n is unstable for all $n \in \mathbb{N}$, since for each n there exists a site x such that $\sigma_n(x) = 4$. However, since each site topples only finitely many times, σ stabilizes.

1.2 Necessary Condition for Stabilizability

Naturally, one may wonder if there are necessary and sufficient conditions for stabilizability of i.i.d. sandpiles.

Proposition 1.1 (Conservation of Density)**.** *Let σ be an i.i.d. sandpile on $\mathbb{Z}^2$ such that $\mathbb{E}[\sigma(x)] = c < \infty$. Furthermore, suppose σ is a.s. stabilizable. Then the expected slope is conserved by the stabilizaton, that is, $\mathbb{E}[\sigma_\infty(x)] = c$.*

Proof. A proof of a more general statement can be found in [2] (Lemma 2.10). □

Lemma 1.1. *An i.i.d. sandpile σ which stabilizes a.s. must satisfy $\mathbb{E}[\sigma(x)] \leq 3$.*

Proof. Clearly a sandpile σ as stated satisfies $\mathbb{E}[\sigma_\infty(x)] \leq 3$. The claim then follows by Proposition 1.1. □

The condition is not sufficient for stabilizability, as shown by the following example.

Example 1.2. *Fix $0 < p \leq 1/2$, then the i.i.d. sandpile defined by*

$$\mathbb{P}[\sigma(x) = 2] = 1 - p$$
$$\mathbb{P}[\sigma(x) = 4] = p$$

a.s. fails to stabilize, despite $2 < \mathbb{E}[\sigma(x)] \leq 3$ [3].

Lemma 1.2. *A sandpile σ such that every site topples at some point of the iterative procedure is not stabilizable.*

Proof. Suppose every site of σ topples. We prove the stronger claim that every site of σ topples infintely often during the iterative procedure.

If this were not the case, some site $x \in \mathbb{Z}^2$ must finish toppling no later than all of its neighbours do, which is aptly known as the *no infinite backward chain condition* (which we shall formally intoduce in the next subsection). Then, since each neighbour topples at least once more, x gains an additional 4 particles and must topple once again, a contradiction. □

Lemma 1.3. *If σ stabilizes, there exists a site x which does not topple at any iteration of the stabilization.*

Proof. This is a simply a reformulation of Lemma 1.2. $\square$

1.3 Toppling Procedures

Considering alternate approaches and characteristics of sandpiles can aid in ones understanding. Some examples of intuitive ways to topple sandpiles are:

- *Topplings in parallel.* Simutaneously topple all unstable sites at discrete time intervals, as decribed earlier by the "iterative procedure".
- *Toppling in nested volumes.* Choose $A_n \subseteq A_{n+1} \subset \mathbb{Z}^2$ such that the union of the subsets is $\mathbb{Z}^2$ and each subset contain finitely many sites. Begin by toppling all unstable sites in A_0 until no unstable sites in A_0 remain, then do the same for A_1, followed by A_2 etc. Similairly to "topplings in parallel", this process is discrete and deterministic.
- *Markov toppling processes.* Each site has a Poisson clock of rate one, which are independent of all other clocks. Assuming a site x is unstable, x topples when the associated clock rings. This is an example of a random toppling procedure.

Definition 1.2. *A (non-random)* toppling procedure *on $\mathbb{Z}^2$ is a measurable map*

$$T : [0, \infty) \times \mathbb{Z}^2 \times \mathbb{N}^{\mathbb{Z}^2} \to \mathbb{N}$$

such that, for all sandpile configurations σ on $\mathbb{Z}^2$, all $x \in \mathbb{Z}^2$,

- *$T(0, x, \sigma) = 0$.*
- *$t \mapsto T(t, x, \sigma)$ is right-continuous, nondecreasing, with jumps of size at most one.*
- *in every finite time interval, there are finitely many jumps at x.*
- *T doesn't contain an 'infinite backward chain of topplings'. That is, there are no infinite chains of topplings at sites $x_1, x_2, ...$ occuring at times $t_1 > t_2 > ...$, where each x_i is a neighbor of x_{i+1}.*

View $T(t, x, \sigma)$ as the number of topplings at site x in the interval $[0, t]$, with initial configuration σ. A toppling procedure T is

- *legal* if only unstable sites are toppled;
- *finite* for initial configuration σ if $T(\infty, x, \sigma) = \lim_{t\to\infty} T(t, x, \sigma) < \infty$, for every site x;

- *stabilizing* for intial configuration σ if it's finite, legal and the resulting configuration σ_∞ is stable.

Theorem 1.1 (Commutativity)**.** *Suppose for a sandpile σ on $\mathbb{Z}^2$ there exists a corresponding stabilizing toppling procedure T.*

Then any *legal toppling procedure for σ is finite. Furthermore if T' is another such stabilizing toppling procedure for σ, then for all sites x,*

$$T(\infty, x, \sigma) = T'(\infty, x, \sigma).$$

Proof. We proceed as in [2], providing more details for the reader where deemed appropriate.

Suppose T, T' are legal toppling procedures which are both finite for initial configuration σ. First we aim to show that if T is stabilizing for σ, then $T'(\infty, x, \sigma) \leq T(\infty, x, \sigma)$, for all $x \in \mathbb{Z}^2$.

Define for each site x an associated time

$$\nu_x = \sup\{t : T'(t, x, \sigma) \leq T(\infty, x, \sigma)\}$$

and call topplings of T' that occur at site x after time ν_x extra topplings. Our aim is then to show that no extra topplings occur, as extra topplings at site x imply $T'(\infty, x, \sigma) > T(\infty, x, \sigma)$.

To derive a contradiction, suppose there exists extra topplings. Let x be such a site, with an extra toppling occuring at time t_x. Then by definition

$$\nu_x < t_x < \infty$$

Let t_x^- denote a time just before t_x. Then it follows $T'(t_x^-, x, \sigma) \geq T(\infty, x, \sigma)$, since toppling procedures are right continuous with jumps of size at most one.

Additionally, for this extra toppling to be legal, site x must be unstable at a time just before t_x. Noting that the expression on the right hand side of (1) is precisely the number of grains at site x, time t_x^- while following procedure T', this can be written as

$$\begin{aligned} 4 &\leq \sigma(x) - 4T'(t_x^-, x, \sigma) + \sum_{y:y\sim x} T'(t_x^-, y, \sigma) \qquad (1) \\ &\leq \sigma(x) - 4T(\infty, x, \sigma) + \sum_{y:y\sim x} T'(t_x^-, y, \sigma). \end{aligned}$$

But since T is stabilizing, we know site x has less than 4 grains of sand under procedure T as $t \to \infty$, i.e.

$$4 > \sigma(x) - 4T(\infty, x, \sigma) + \sum_{y:y\sim x} T(\infty, y, \sigma).$$

Combining the two inequalities tells us that for at least one neighbour y of x, $T'(t_x^-, y, \sigma) > T(\infty, y, \sigma)$, i.e. for the extra toppling at site x to be legal it must be preceeded by an extra toppling at one of its neighbours. We can then argue similairly to prove the neighbour is also preceeded by another extra toppling at a neighbouring site, and so on.

Continuing, a legal extra toppling at site x implies the existence of an infinite backwards chain of topplings, contradicting the definition of a toppling procedure. Thus, no extra topplings exist and consequently $T'(\infty, x, \sigma) \leq T(\infty, x, \sigma)$ is satisfied for all $x \in \mathbb{Z}^2$. Immediately, we can then deduce that if both T, T' are stabilizing toppling procedures for σ, then $T'(\infty, x, \sigma) = T(\infty, x, \sigma)$ for all $x \in \mathbb{Z}^2$. $\square$

Remark The theorem implies the final configuration of a stabilizable sandpile is irrespective of toppling procedure. Further, it says the number of topplings at each site is also irrespective of the toppling procedure, meaning our earlier definitions of σ_∞ and stabilization are well defined beyond the context of the "topplings in parallel" procedure.

2 Greens Function and Derivatives

The following theorem is the main concern of the next section, although we do not yet have the tools to prove it, hopefully some motivation is provided with regards to the results presented in this section.

Recall that if σ is an i.i.d. sandpile which stabilizes almost surely, then $\mathbb{E}[\sigma(x)] \leq 3$.

Theorem 2.1. *There are constants $c, d > 0$ such that any i.i.d. sandpile σ on $\mathbb{Z}^2$ that stabilizes almost surely satisfies*

$$\mathbb{E}[\sigma(x)] \leq 3 - min(c, d\mathbb{E}[|X - X'|^{2/3}])$$

for X, X' independent and distributed as $\sigma(x)$.

We may interpret the theorem as follows:

- Suppose we have a sandpile σ with expected height slightly less than 3, more preciesly, suppose it satisfies $3 - c < \mathbb{E}[\sigma(x)] \leq 3$. Then σ cannot stabilize a.s. unless $\mathbb{E}[|X - X'|^{2/3}]$ is very small.
- $\mathbb{P}[X \neq X'] \leq \mathbb{E}[|X - X'|^{2/3}]$ implies that if σ stabilizes almost surely and has expected height slightly less than 3, then as a result of the observation above, the law of σ is concentrated at a single value. It's clear if this is the case, the value which σ is concetrated at is at most 3.

2.1 Functions on the Square Lattice

Harmonicity modulo 1 of functions, which we shall define shortly, is a crucial property required for the proof of Theorem 2.1.

Definition 2.1. *Let $G = (V, E)$ be a graph. Let $\psi : V \to R$ be a function of the vertices taking values in a ring. Then, the discrete Laplacian Δ acting on ψ is defined by*

$$(\Delta\psi)(v) = \sum_{u:u\sim v} (\psi(v) - \psi(u))$$

where $u \sim v$ denotes that $u, v \in V$ are neighbours.

An elementary example of the Laplace operator is now shown.

Example 2.1. *Consider the graph in figure 1. Suppose ψ is a function on the vertices which counts the number of neighbours, then,*

$$(\Delta\psi)(A) = \sum_{u:u\sim A} (\psi(A) - \psi(u)) = 2\psi(A) - \psi(B) - \psi(C) = -1.$$

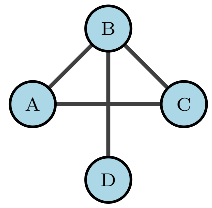

Figure 1: *A simple, undirected graph.*

Definition 2.2 (Harmonic modulo one)**.** *Let $f : \mathbb{Z}^2 \to \mathbb{C}$ be a function. Say f is harmonic modulo one if the discrete Laplacian of f is integer valued, i.e.*

$$(\Delta f)(x) = 4f(x) - \sum_{y:y\sim x} f(y) \in \mathbb{Z}, \textit{ for all } x \in \mathbb{Z}^2.$$

Fix $1 \leq p < \infty$. Recall that a function $f : X \to \mathbb{C}$ (on a countable domain X) is in $\ell^p(X)$ if $\sum_{x\in X}|f(x)|^p < \infty$.

Definition 2.3. *For $1 \leq p < \infty$, we define,*

$$\mathscr{H}^p(\mathbb{Z}^2) = \{f \in \ell^p(\mathbb{Z}^2) \mid f \textit{ is harmonic modulo one}\}.$$

Additionally, we adopt the standard definition for the support of a function $f : X \to \mathbb{C}$ given by,

$$supp(f) = \{x \in X : f(x) \neq 0\}.$$

2.2 Greens Function

Let $p_n(x)$ denote the the probability of a simple symmetric random walk on $\mathbb{Z}^2$ landing on site x at time n. We now define a function possessing some of the properties we recently introduced.

Definition 2.4. *Greens function on $\mathbb{Z}^2$ is defined by,*

$$G_{\mathbb{Z}^2}(x) = \frac{1}{4}\sum_{n=0}^{\infty}[p_n(x) - p_n(0)].$$

Equivalently, $4G_{\mathbb{Z}^2}(x)$ is the expected number of visits to site x, minus the expected number of visits to the origin, up to time N, as $N \to \infty$. In related texts, Greens function is often called the 'potential kernel' and denoted by $a(x)$.

We now state asymptotics regarding the transition probability $p_{2n}(0)$ on $\mathbb{Z}^2$, which will prove useful shortly.

Proposition 2.1. *For a simple symmetric random walk on* $\mathbb{Z}^2$*,*

$$p_{2n}(0) \sim \frac{2}{A^2 n}, \text{ as } n \to \infty.$$

for some constant $A > 0$*.*

Proof. This is proved in [4] (Example 1.6.2). □

Theorem 2.2. *Greens function is finite, i.e., G(x) is well-defined.*

Proof. Proceeding as in [5] with some additional detail, we first notice that $G(0) = 0$.

For $x, y \in \mathbb{Z}^2$, write $x \leftrightarrow y$ if the verticies have the same parity, i.e. $x_1 + x_2 + y_1 + y_2$ is even. Picturing $\mathbb{Z}^2$ as an infinite chess board, parity is denoted by the colour of the square. Similairly, for $n \in \mathbb{Z}$, write $x \leftrightarrow n$ if $x_1 + x_2 + n$ is even.

We introduce an estimate for p_n. Define $\bar{p}_0(x) = \mathbb{1}\{x = 0\}$ and for $n \geq 1$,

$$\bar{p}_n(x) = \frac{2}{\pi n} \cdot \exp\left(-\frac{|x|^2}{n}\right)$$

also, define the error $E_n(x)$ by

$$E_n(x) = \begin{cases} p_n(x) - \bar{p}_n(x) & \text{if } n \leftrightarrow x \\ 0 & \text{if } n \not\leftrightarrow x \end{cases}$$

Then for each $n \in \mathbb{N}$ and $x \neq 0$, $x \leftrightarrow 0$, using the standard inequality $e^x \geq 1 + x$ for all $x \in \mathbb{R}$,

$$\begin{aligned} \bar{p}_{2n}(0) - \bar{p}_{2n}(x) &= \frac{1}{\pi n}\left[1 - \exp\left(-\frac{|x|^2}{2n}\right)\right] \\ &\leq \frac{|x|^2}{2\pi n^2} \end{aligned}$$

thus,

$$\sum_{n=1}^{\infty} |\bar{p}_{2n}(x) - \bar{p}_{2n}(0)| < \infty.$$

Noting that for n odd $p_n(0) = p_n(x) = 0$, we may write

$$\begin{aligned} \sum_{n=0}^{j} p_n(x) - p_n(0) &= 1 + \sum_{1 \leq n \leq \frac{j}{2}} p_{2n}(x) - p_{2n}(0) \\ &= 1 + \sum_{1 \leq n \leq \frac{j}{2}} [\bar{p}_{2n}(x) - \bar{p}_{2n}(0)] + \sum_{n=1}^{j} E_n(x) - \sum_{n=1}^{j} E_n(0). \quad (2) \end{aligned}$$

By the 'local central limit theorem' provided in [5] (specifically, Theorem 1.2.1), $|E_n(x)| \leq O(n^{-2})$, thus

$$\sum_{n=1}^{\infty} |E_n(x)| < \infty, \quad \sum_{n=1}^{\infty} |E_n(0)| < \infty.$$

By letting $j \to \infty$ in (2), conclude that $G(x)$ is indeed finite for such x.

Now we consider $x \in \mathbb{Z}^2$ such that $x \neq 0$, $x \not\leftrightarrow 0$.

Firstly, by the law of total probability, $p_{n+1}(x) = \frac{1}{4} \sum_{y:y\sim x} p_n(y)$ for every $n \in \mathbb{N}_0$, $x \in \mathbb{Z}^2$, since the random walk must visit from a neighbouring site. Thus, noting that $p_0(x) = 0$ since $x \neq 0$,

$$\sum_{n=0}^{N} p_n(x) = \frac{1}{4} \sum_{y:y\sim x} \left[\sum_{n=0}^{N-1} p_n(y) \right].$$

Implying that

$$\begin{aligned} \sum_{n=0}^{N} [p_n(x) - p_n(0)] &= \frac{1}{4} \sum_{y:y\sim x} \left[\sum_{n=0}^{N-1} p_n(y) - p_n(0) \right] - p_N(0) \\ &\to \sum_{y:y\sim x} G(y), \text{ as } N \to \infty, \end{aligned}$$

where we justify $p_N(0) \to 0$ by Proposition 2.1.

Lastly, since every neighbour of x must have the same parity as 0, we can apply the last case and deduce the result. $\square$

Greens function will prove essential in characterising the functions in $\mathscr{H}^1(\mathbb{Z}^2)$, the purpose of which shall become apparent as we work towards proving theorem 1, but for this to be possible some work is required.

Definition 2.5. *The discrete derivatives of a function $f : \mathbb{Z}^2 \to \mathbb{C}$ are*

$$\begin{aligned} D_1 f(i,j) &\coloneqq f(i+1,j) - f(i,j), \\ D_2 f(i,j) &\coloneqq f(i,j+1) - f(i,j). \end{aligned}$$

Lemma 2.1. *$G_{\mathbb{Z}^2}$ and its discrete derivatives are harmonic modulo one.*

Proof. Using $p_{n+1}(x) = \frac{1}{4}\sum_{y:y\sim x} p_n(y), n \geq 0$,

$$\begin{aligned}
(\Delta G_{\mathbb{Z}^2})(x) &= 4G_{\mathbb{Z}^2}(x) - \sum_{y:y\sim x} G_{\mathbb{Z}^2}(y) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \left[p_n(x) - \frac{1}{4}\sum_{y:y\sim x} p_n(y) \right] \\
&= \lim_{N\to\infty} \left[p_0(x) - \frac{1}{4}\sum_{y:y\sim x} p_N(y) \right] \\
&= p_0(x) - \frac{1}{4}\sum_{y:y\sim x} \lim_{N\to\infty} p_N(y) \\
&= p_0(x) = \mathbb{1}\{x = (0,0)\}
\end{aligned}$$

by null recurrence. Thus, $G_{\mathbb{Z}^2}$ is harmonic modulo one. It's easy to see that any translates of harmonic modulo one functions, including Greens discrete derivatives, are also harmonic modulo one. □

2.3 Characterisation of $\mathscr{H}^1(\mathbb{Z}^2)$

For functions $f_1, ..., f_n$ on $\mathbb{Z}^2$, let $\langle\langle f_1, ..., f_n \rangle\rangle$ be the finite integer span of translates of f. More formally,

$$\langle\langle f_1, ..., f_n \rangle\rangle \coloneqq \text{span}_{\mathbb{Z}}\{T_x f_1, ..., T_x f_n : x \in \mathbb{Z}^2\},$$

where T is the translation operator, defined by $T_{(i,j)} f(x, y) \coloneqq f(x - i, y - j)$ for $(i, j) \in \mathbb{Z}^2$.

Example 2.2. *We claim that* $\langle\langle \mathbb{1}\{x = (0,0)\} \rangle\rangle = \{f : \mathbb{Z}^2 \to \mathbb{Z} \mid ||f||_1 < \infty\}$.

- *It's clear the left hand side is contained in the right hand side, since* $T_{(i,j)}\mathbb{1}\{x = (0,0)\} = \mathbb{1}\{x = (i,j)\}$, *and every function in the finite integer span of indicators is necessarily in* $\ell^1(\mathbb{Z}^2)$ *and integer valued.*
- *Let* f *belong to the set on the right hand side. Then, since* f *is integer valued and satisfies* $\sum_{x\in\mathbb{Z}^2} |f(x)| < \infty$, *it must have finite support and* $|f| < \infty$ *pointwise. Thus, it must be in the finite integer span of indicator functions.*

It's known classically that for $x = (x_1, x_2)$ with norm $||x||_2 = \sqrt{x_1^2 + x_2^2}$, that $G_{\mathbb{Z}^2}(x) = -\frac{1}{2\pi}\log||x||_2 + O(1)$. A more precise estimation is now introduced.

Theorem 2.3. *Let* $x = (x_1, x_2) \in \mathbb{Z}^2 \setminus (0,0)$. *There exists* $a, b > 0$ *such that*

$$G_{\mathbb{Z}^2}(x) = -\frac{1}{2\pi} log||x||_2 - a - \frac{b}{||x||_2^2}\left(\frac{8x_1^2 x_2^2}{||x||_2^4} - 1\right) + O\left(||x||_2^{-4}\right).$$

Proof. The proof is provided in [1] (Theorem 6). ∎

Lemma 2.2. *For* $(x_1, x_2) \in \mathbb{Z}^2 \setminus (0,0)$*, we have:*

$$\begin{aligned}
D_1 G_{\mathbb{Z}^2}(x_1,x_2) &= \frac{1}{2\pi} \cdot \frac{-x_1}{||x||_2^2} + O\left(||x||_2^{-2}\right) \\
D_2 G_{\mathbb{Z}^2}(x_1,x_2) &= \frac{1}{2\pi} \cdot \frac{-x_2}{||x||_2^2} + O\left(||x||_2^{-2}\right) \\
D_1^2 G_{\mathbb{Z}^2}(x_1,x_2) &= \frac{1}{2\pi} \cdot \frac{x_1^2 - x_2^2}{||x||_2^4} + O\left(||x||_2^{-3}\right) \\
D_2^2 G_{\mathbb{Z}^2}(x_1,x_2) &= \frac{1}{2\pi} \cdot \frac{x_2^2 - x_1^2}{||x||_2^4} + O\left(||x||_2^{-3}\right) \\
D_1 D_2 G_{\mathbb{Z}^2}(x_1,x_2) &= \frac{1}{2\pi} \cdot \frac{2x_1x_2}{||x||_2^4} + O\left(||x||_2^{-3}\right) \\
D_1^a D_2^b G_{\mathbb{Z}^2}(x_1,x_2) &= O\left(||x||_2^{-3}\right), a+b=3.
\end{aligned}$$

Proof. For brevity, we shall only prove the first case. The remaining statements are proved analogously.

Let $x \in \mathbb{Z}^2 \setminus \{(0,0),(-1,0)\}$, then using the estimation in Theorem 2.3,

$$\begin{aligned}
D_1 G_{\mathbb{Z}^2}(x_1,x_2) &= G_{\mathbb{Z}^2}(x_1+1,x_2) - G_{\mathbb{Z}^2}(x_1,x_2) \\
&= -\frac{1}{2\pi} \log \frac{\sqrt{(x_1+1)^2 + x_2^2}}{\sqrt{x_1^2+x_2^2}} - \frac{b}{(x_1+1)^2+x_2^2}\left(\frac{8(x_1+1)^2x_2^2}{((x_1+1)^2+x_2^2)^2} - 1\right) \\
&+ \frac{b}{x_1^2+x_2^2}\left(\frac{8x_1^2x_2^2}{(x_1^2+x_2^2)^2} - 1\right) + O\left(||x||_2^{-4}\right) \\
&= -\frac{1}{4\pi}\log\left(1 + \frac{2x_1+1}{x_1^2+x_2^2}\right) + b\left(\frac{1}{(x_1+1)^2+x_2^2} - \frac{1}{x_1^2+x_2^2}\right) \\
&- 8b\left(\frac{(x_1+1)^2x_2^2}{((x_1+1)^2+x_2^2)^3} - \frac{x_1^2x_2^2}{(x_1^2+x_2^2)^3}\right) + O\left(||x||_2^{-4}\right)
\end{aligned}$$

By using a common denominator, it follows that the terms in brackets with coefficient b are $O\left(||x||_2^{-3}\right)$. Now, expand the log term using the following Taylor series,

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - ..., \text{ for } x \in (0,1].$$

We calculate

$$
\begin{aligned}
-\frac{1}{4\pi}\log\left(1+\frac{2x_1+1}{x_1^2+x_2^2}\right) &= -\frac{1}{4\pi}\left[\frac{2x_1+1}{x_1^2+x_2^2}-\frac{1}{2}\frac{(2x_1+1)^2}{(x_1^2+x_2^2)^2}\right]+O\left(||x||_2^{-3}\right)\\
&= -\frac{1}{2\pi}\cdot\frac{x_1}{x_1^2+x_2^2}-\frac{1}{4\pi}\left[\frac{1}{x_1^2+x_2^2}-\frac{2x_1^2}{(x_1^2+x_2^2)^2}\right]+O\left(||x||_2^{-3}\right)\\
&= -\frac{1}{2\pi}\cdot\frac{x_1}{x_1^2+x_2^2}-\frac{1}{4\pi}\cdot\frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}+O\left(||x||_2^{-3}\right).
\end{aligned}
$$

Thus, we see that $D_1 G_{\mathbb{Z}^2}(x_1,x_2)=\frac{1}{2\pi}\cdot\frac{-x_1}{||x||_2^2}+O\left(||x||_2^{-2}\right)$, as stated.

□

Theorem 2.4 (Characterisation of $\mathscr{H}^1(\mathbb{Z}^2)$)**.** *We claim that,*

$$\mathscr{H}^1(\mathbb{Z}^2)=\langle\langle D_1^3G_{\mathbb{Z}^2}, D_1^2D_2G_{\mathbb{Z}^2}, D_1D_2^2G_{\mathbb{Z}^2}, D_2^3G_{\mathbb{Z}^2}, \mathbb{1}\{x=(0,0)\}\rangle\rangle.$$

Proof. Here we shall only show that the right hand side is contained in the left hand side,

- It's clear $\mathbb{1}\{x=(0,0)\}$ is harmonic modulo one, furthermore, the functions

  $$D_1^3G_{\mathbb{Z}^2}, D_1^2D_2G_{\mathbb{Z}^2}, D_1D_2^2G_{\mathbb{Z}^2}, D_2^3G_{\mathbb{Z}^2}$$

  are also harmonic modulo one by Lemma 2.1. It's apparent that harmonicity modulo one is preserved by translation and also preserved under linear combinations which are finite and integer valued. Thus, functions on the right hand side are harmonic modulo one.

- Additionally, we must show functions on the right hand side are in $\ell^1(\mathbb{Z}^2)$. First, for every $k\in\mathbb{N}_0$, define

  $$R_k=\{x\in\mathbb{Z}^2: 2^k\le ||x||_2<2^{k+1}\}=B_{2^{k+1}}(0)\setminus B_{2^k}(0),$$

  so $\{R_k : k\in\mathbb{N}_0\}$ partitions $\mathbb{Z}^2\setminus\{0\}$ and $|R_k|\le(2^{k+2})^2$, since R_k is contained in the open square of side length 2^{k+2} centered at the origin, which contains $(2^{k+2}-1)^2$ points.

  Let $f:\mathbb{Z}^2\to\mathbb{R}$ be a function such that $f=O\left(||x||_2^{-3}\right)$. Then for some constant $C > 0$,

$$\begin{aligned}
\sum_{x \in \mathbb{Z}^2} |f(x)| &\leq \sum_{x \in \mathbb{Z}^2 \setminus \{0\}} C \cdot \frac{1}{||x||_2^3} \\
&= C \sum_{k=0}^{\infty} \sum_{x \in R_k} \frac{1}{||x||_2^3} \\
&\leq C \sum_{k=0}^{\infty} \sum_{x \in R_k} \frac{1}{(2^k)^3} \\
&\leq C \sum_{k=0}^{\infty} \frac{(2^{k+2})^2}{(2^k)^3} \\
&= 16C \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty,
\end{aligned}$$

implying $f \in \ell^1(\mathbb{Z}^2)$.

By Lemma 2.2, third discrete derivatives of Greens function are $O\left(||x||_2^{-3}\right)$, so they're in $\ell^1(\mathbb{Z}^2)$. Finally, the space $\ell^1(\mathbb{Z}^2)$ is closed under translation and finite integer span, which completes the claim.

This shows the right hand side is contained in the left hand side. The other inclusion is proved in [1] (Theorem 4). $\square$

3 Proving the Upper Bound

Recall that given a sandpile $\sigma : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$, the *parallel toppling procedure* gives a sequence of sandpiles $\sigma_0, \sigma_1, \sigma_2, ...$ where σ_{n+1} is obtained from σ_n by simultaneously toppling each site x such that $\sigma_n(x) \geq 4$. We now introduce some formal notation.

Definition 3.1 (Topplings in Parallel)**.** *Let $\sigma = \sigma_0$ be a sandpile on $\mathbb{Z}^2$. Define sandpiles $(\sigma_n)_{n\in\mathbb{N}_0}$ recursively as follows,*

$$\sigma_{n+1} = \sigma_n - \Delta(v^n) = \sigma - \Delta(u^{n+1}),$$

where the functions v^n are defined by

$$v^n(x) := \mathbb{1}\{\sigma_n(x) \geq 4\}, n \in \mathbb{N}_0$$

and the odometer functions, u^n, are given by

$$u^n(x) := v^0 + v^1 + ... + v^{n-1}, n \in \mathbb{N}.$$

Notice that $u^n(x)$ is the number of times site x has toppled in the first n topplings.

Lemma 3.1. *A sandpile σ stabilizes if and only if $u^n \uparrow u^\infty$ for some function $u^\infty : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$. In this case, $\sigma_\infty = \sigma - \Delta(u^\infty)$.*

Proof. We prove contrapositives,

- If σ doesn't stabilize, some site $x \in \mathbb{Z}^2$ must topple a non-finite number of times throughout the parallel topplings procedure. In this case, $u^n(x)$ diverges as $n \to \infty$.
- Similairly, if u^n does not converge then for some site x we have $u^n(x) \to \infty$, as $n \to \infty$, implying the site topples infintely often and σ does not stabilize.

□

3.1 Properties of Functions in $\mathscr{H}^1(\mathbb{Z}^2)$

Now, consider the pairing

$$\langle \sigma, \xi \rangle = \sum_{x\in\mathbb{Z}^2} \sigma(x)\xi(x).$$

We claim that if $\xi \in \mathscr{H}^1(\mathbb{Z}^2)$, then the pairing is invariant modulo one when the sandpile σ is stabilized. Before we prove this however, some preliminary results are required.

Lemma 3.2. *Let $\xi \in \mathscr{H}^1(\mathbb{Z}^2)$. Then $\Delta\xi$ has finite support.*

Proof. First we show $\Delta\xi \in \ell^1(\mathbb{Z}^2)$,

$$\begin{aligned}
\sum_{x\in\mathbb{Z}^2} |\Delta\xi(x)| &= \sum_{x\in\mathbb{Z}^2} |4\xi(x) - \sum_{y:x\sim y} \xi(y)| \\
&\leq \sum_{x\in\mathbb{Z}^2} \left[4|\xi(x)| + \sum_{y:x\sim y} |\xi(y)| \right] \\
&= 4||\xi||_1 + \sum_{x\in\mathbb{Z}^2} \sum_{y:x\sim y} |\xi(y)| \\
&\leq 8||\xi||_1 < \infty.
\end{aligned}$$

Also, $\Delta\xi$ is integer valued since ξ is harmonic modulo one, so the absolute convergence implies finite support. □

Lemma 3.3. *The discrete Laplacian Δ is a self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ equipped with the pairing $\langle \cdot, \cdot \rangle$.*

Proof. Let $f, g \in \ell^2(\mathbb{Z}^2)$, then

$$\begin{aligned}
\langle f, \Delta g \rangle &= \sum_{x\in\mathbb{Z}^2} f(x) \cdot (\Delta g)(x) \\
&= \sum_{x\in\mathbb{Z}^2} f(x) \cdot \left[4g(x) - \sum_{y:y\sim x} g(y) \right] \\
&= \sum_{y\in\mathbb{Z}^2} 4f(y)g(y) - \sum_{x\in\mathbb{Z}^2} \sum_{y:y\sim x} f(x)g(y) \\
&= \sum_{y\in\mathbb{Z}^2} 4f(y)g(y) - \sum_{y\in\mathbb{Z}^2} \sum_{x:x\sim y} f(x)g(y) \\
&= \sum_{y\in\mathbb{Z}^2} g(y) \cdot \left[4f(y) - \sum_{x:x\sim y} f(x) \right] \\
&= \langle \Delta f, g \rangle
\end{aligned}$$

□

Lemma 3.4. *Let σ be an i.i.d. sandpile which stabilizes almost surely and let $\xi \in \mathscr{H}^1(\mathbb{Z}^2)$. Then $\langle \sigma, \xi \rangle \equiv \langle \sigma_\infty, \xi \rangle$ mod 1, almost surely.*

Proof. We proceed as in [1], with additional details supplemented. By conservation of density, $\mathbb{E}[\sigma(x)] \leq 3$, then

$$\begin{aligned}\mathbb{E}\left[\langle\sigma, |\xi|\rangle\right] &= \mathbb{E}\left[\sum_{x\in\mathbb{Z}^2} \sigma(x)\cdot|\xi(x)|\right] \\ &\overset{\text{MON}}{=} \sum_{x\in\mathbb{Z}^2} \mathbb{E}\left[\sigma(x)\cdot|\xi(x)|\right] \\ &= \sum_{x\in\mathbb{Z}^2} |\xi(x)|\cdot\mathbb{E}\left[\sigma(x)\right] \\ &= ||\xi||_1\mathbb{E}[\sigma_0] < \infty\end{aligned}$$

where we use monotone convergence for non-negative sums in the second equality and linearity in the third (noting that ξ is not a random variable),

$$\Rightarrow \langle\sigma, \xi\rangle \text{ converges absolutely, almost surely.}$$

Now, as before write $\sigma_n = \sigma - \Delta u^n$, and by self-adjointness of Δ,

$$\langle\sigma_n, \xi\rangle = \langle\sigma - \Delta u^n, \xi\rangle = \langle\sigma, \xi\rangle - \langle u^n, \Delta\xi\rangle. \tag{3}$$

By Lemma 3.2, $\Delta\xi$ has finite support S, and as a result,

$$\langle u^n, \Delta\xi\rangle = \sum_{x\in S}\Delta\xi(x)\cdot u^n(x) \to \sum_{x\in S}\Delta\xi(x)\cdot u^\infty(x) = \langle u^\infty, \Delta\xi\rangle \in \mathbb{Z} \tag{4}$$

where the converegence follows from $|S| < \infty$, as well as $u^n \uparrow u^\infty$. Additionally, the sum on the right hand side is a finite sum of integers, implying $\langle u^\infty, \Delta\xi\rangle$ is finite and moreover an integer.

Notice that due to our topplings in parallel procedure, the following inequality is satisfied for all $n \geq 0$,

$$\sigma_{n+1}(x) \leq \max(\sigma_n(x), 7).$$

To see this consider two cases,

- If $\sigma_n(x) \leq 3$, then $\sigma_{n+1}(x) \leq 7$ since it gains at most 4 particles from its neighbours (one from each).
- If $\sigma_n(x) \geq 4$, then $\sigma_{n+1}(x) \leq \sigma_n(x)$ since it must topple and lose 4 particles however it gains at most 4 particles from its neighbours.

Thus, in the interest of applying dominated convergence, notice that whenever $\langle\sigma, |\xi|\rangle < \infty$ and σ stabilizes to σ_∞,

$$|\sigma_n(x)\xi(x)| \leq \max(\sigma(x), 7)\cdot|\xi(x)|, \quad \sum_{x\in\mathbb{Z}^2}\max(\sigma(x), 7)\cdot|\xi(x)| < \infty. \tag{5}$$

So, using (5) to apply dominated convergence in the second equality below,

$$\begin{aligned}\lim_{n\to\infty}\langle\sigma_n,\xi\rangle = \lim_{n\to\infty}\sum_{x\in\mathbb{Z}^2}\sigma_n(x)\xi(x) &\overset{\mathsf{DOM}}{=} \sum_{x\in\mathbb{Z}^2}\lim_{n\to\infty}\sigma_n(x)\xi(x)\\ &= \langle\sigma_\infty,\xi\rangle \end{aligned}\tag{6}$$

Considering (3) as $n\to\infty$ then yields the desired result. More explicitly,

$$\begin{aligned}\langle\sigma_\infty,\xi\rangle &= \langle\sigma,\xi\rangle - \underbrace{\langle u^\infty,\Delta\xi\rangle}_{\in\,\mathbb{Z}}, \text{ by (4), (6),}\\ \Rightarrow \langle\sigma_\infty,\xi\rangle &\equiv \langle\sigma,\xi\rangle \bmod 1, \text{ almost surely.}\end{aligned}$$

□

We now provide a definition and an elementary inequality which will prove useful later.

Definition 3.2 (Complex RV)**.** *A complex random variable Z on a probability space is a function such that both its real part $\Re(Z)$ and its imaginary part $\Im(Z)$ are real random variables on the probability space. Furthermore, define the expectation as $\mathbb{E}[Z]=\mathbb{E}[\Re(Z)]+i\mathbb{E}[\Im(Z)]$.*

Lemma 3.5. *For all $t\in\mathbb{R}$, $|1-e^{2\pi it}|\le 2\pi|t|$.*

Proof. Let $t\in\mathbb{R}$. The expression $1-e^{2\pi it}$ can be written as

$$1-e^{2\pi it} = -\int_0^t 2\pi i\cdot e^{2\pi is}\,ds$$

Thus, for $t\ge 0$, we can bound the expression as follows

$$\begin{aligned}|1-e^{2\pi it}| &\le \int_0^t \underbrace{|2\pi i|}_{=\,2\pi}\cdot\underbrace{|e^{2\pi is}|}_{=\,1}\,ds\\ &= \int_0^t 2\pi\,ds\\ &= 2\pi t.\end{aligned}$$

Similairly, if $t<0$, we obtain $|1-e^{2\pi it}|\le -2\pi t$. □

Throughout the proof of the upper bound we require a function from the set $\mathscr{H}^1(\mathbb{Z}^2)$, so for definiteness and simplicity we use $\xi = D_1^3 G_{\mathbb{Z}^2}$, which is in $\mathscr{H}^1(\mathbb{Z}^2)$ by Theorem 2.4. Now we prove a final lemma before we get to the main theorem.

Lemma 3.6. *Let $R\ge 1$ be a parameter. Then, as $R\to\infty$,*

$$\sum_{x\in\mathbb{Z}^2:0<|\xi(x)|<\frac{1}{2R}}|\xi(x)|^2 \gg R^{-\frac{4}{3}}.$$

Proof. Once again, proceeding as in [1], we begin by showing that there exists $0 \leq \theta_1 < \theta_2 < 2\pi$ such that

$$||x||_2^{-3} \gg |\xi(x)| \gg ||x||_2^{-3}$$

for all $x \in \mathbb{Z}^2 \setminus \{(0,0)\}$ satisfying $\theta_1 \leq \arg(x) \leq \theta_2$. By Lemma 2.2 we have that $D_1^3 G_{\mathbb{Z}^2} = O(||x||_2^{-3})$, so it just remains to show $|\xi(x)| \gg ||x||_2^{-3}$ for points x lying in such a segment.

Recall from the proof of Lemma 2.2 that

$$\begin{aligned} D_1 G(x_1, x_2) = &-\frac{1}{4\pi} \log \left(1 + \frac{2x_1 + 1}{x_1^2 + x_2^2}\right) + b \left(\frac{1}{(x_1+1)^2 + x_2^2} - \frac{1}{x_1^2 + x_2^2}\right) \\ &- 8b \left(\frac{(x_1+1)^2 x_2^2}{((x_1+1)^2 + x_2^2)^3} - \frac{x_1^2 x_2^2}{(x_1^2 + x_2^2)^3}\right) + O\left(\frac{1}{||x||_2^4}\right). \end{aligned}$$

Then we proceed as follows,

$$\begin{aligned} D_1^2 G(x_1, x_2) &= D_1 G(x_1 + 1, x_2) - D_1 G(x_1, x_2) \\ &= -\frac{1}{4\pi} \left[\log \left(1 + \frac{2(x_1+1)+1}{(x_1+1)^2 + x_2^2}\right) - \log \left(1 + \frac{2x_1 + 1}{x_1^2 + x_2^2}\right)\right] \\ &\quad + O(||x||_2^{-4}) \\ &= -\frac{1}{4\pi} \log \left(\frac{[(x_1+2)^2 + x_2^2] \cdot [x_1^2 + x_2^2]}{[(x_1+1)^2 + x_2^2]^2}\right) + O(||x||_2^{-4}). \end{aligned} \tag{7}$$

However, we have not yet justified that the bracketed terms of $D_1 G$ which are $O(||x||_2^{-3})$ become $O(||x||_2^{-4})$ after taking the second discrete derivative of G. We now prove this assertion for the first bracketed term by showing $\psi(x_1, x_2)$ is $O(||x||_2^{-4})$, where ψ is defined as follows,

$$\left(\frac{1}{(x_1+2)^2 + x_2^2} - \frac{1}{(x_1+1)^2 + x_2^2}\right) - \left(\frac{1}{(x_1+1)^2 + x_2^2} - \frac{1}{x_1^2 + x_2^2}\right) = \psi(x_1, x_2).$$

Firstly,

$$\begin{aligned} \frac{1}{(x_1+1)^2 + x_2^2} &= \frac{1}{x_1^2 + 2x_1 + x_2^2 + 1} \\ &= \frac{1}{x_1^2 + x_2^2} \frac{1}{1 + \frac{2x_1+1}{x_1^2 + x_2^2}} \\ &= \frac{1}{x_1^2 + x_2^2} \left[1 - \frac{2x_1 + 1}{x_1^2 + x_2^2} + O(||x||_2^{-2})\right]. \end{aligned}$$

Similairly,

$$\frac{1}{(x_1-1)^2 + x_2^2} = \frac{1}{x_1^2 + x_2^2} \left[1 - \frac{-2x_1 + 1}{x_1^2 + x_2^2} + O(||x||_2^{-2})\right].$$

Combining, we find

$$\begin{aligned}\psi(x_1-1,x_2) &= \frac{1}{(x_1+1)^2+x_2^2} - \frac{2}{x_1^2+x_2^2} + \frac{1}{(x_1-1)^2+x_2^2} \\ &= \frac{1}{x_1^2+x_2^2}\left[-\frac{2}{x_1^2+x_2^2} + O(||x||_2^{-2})\right] \\ &= O(||x||_2^{-4}).\end{aligned}$$

Thus we conclude that $\psi(x_1,x_2) = O(||x||_2^{-4})$. Performing a similair verification for the discrete derivative of the other bracketed term in D_1G confirms the accuracy of (7).

Continuing, calculating $\xi(x)$, once again disregarding the low order terms,

$$\begin{aligned}\xi(x_1,x_2) &= D_1^2G(x_1+1,x_2) - D_1^2G(x_1,x_2) \\ &= -\frac{1}{4\pi}\log\left(\frac{[(x_1+3)^2+x_2^2]\cdot[(x_1+1)^2+x_2^2]}{[(x_1+2)^2+x_2^2]^2}\right) \\ &+ \frac{1}{4\pi}\log\left(\frac{[(x_1+2)^2+x_2^2]\cdot[x_1^2+x_2^2]}{[(x_1+1)^2+x_2^2]^2}\right) + O(||x||_2^{-4}) \\ &= -\frac{1}{4\pi}\log\left(\frac{[(x_1+3)^2+x_2^2]\cdot[(x_1+1)^2+x_2^2]^3}{[(x_1+2)^2+x_2^2]^3\cdot[x_1^2+x_2^2]}\right) + O(||x||_2^{-4})\end{aligned}$$

Then, switching to polar co-ordinates by substituting in

$$r = \sqrt{x_1^2+x_2^2} = ||x||_2, \quad x_1 = r\cos\theta$$

we obtain,

$$\xi(r,\theta) = -\frac{1}{4\pi}\log\underbrace{\left(\frac{[r^2+2r\cos\theta+1]^3\cdot[r^2+6r\cos\theta+9]}{r^2[r^2+4r\cos\theta+4]^3}\right)}_{=A} + O(r^{-4}).$$

In the interest of expanding the log term we also calculate

$$A-1 = \frac{16r^5(\cos\theta)^3 - 12r^5\cos\theta}{r^2(r^2+4r\cos\theta+4)^3} + O(r^{-4})$$

then, expanding yields

$$\begin{aligned}4\pi\cdot|\xi(r,\theta)| &= |(A-1) - \frac{(A-1)^2}{2} + ...| + O(r^{-4}) \\ &= |\frac{16r^5(\cos\theta)^3 - 12r^5\cos\theta}{r^2(r^2+4r\cos\theta+4)^3}| + O(r^{-4}).\end{aligned}$$

Finally, for θ around $\frac{2\pi}{3}$,

$$4\pi \cdot r^3 \cdot |\xi(r,\theta)| = \frac{16r^6(\cos\theta)^3 - 12r^6\cos\theta}{(r^2+4r\cos\theta+4)^3} + O(r^{-1}) > 4$$

which proves $|\xi(x)| \gg ||x||_2^{-3}$, for x chosen such that $\arg(x)$ lies in a small half-open interval about $\frac{2\pi}{3}$. Thus using polar co-ordinates we derive that

$$\sum_{x\in\mathbb{Z}^2:0<|\xi(x)|<\frac{1}{2R}} |\xi(x)|^2 \gg \int_{R^{\frac{1}{3}}}^{\infty} \frac{1}{r^5}\, dr \gg R^{-\frac{4}{3}}.$$

$\square$

3.2 Completing the Proof

We're now equipped to prove the upper bound. For convenience, we state it again.

Theorem 3.1. *There are constants $c, d > 0$ such that any i.i.d. sandpile σ on $\mathbb{Z}^2$ that stabilizes almost surely satisfies*

$$\mathbb{E}[\sigma(x)] \leq 3 - min(c, d\mathbb{E}[|X - X'|^{2/3}]) \tag{8}$$

Proof. As in [1], introduce the following characteristic functions,

$$\chi(\sigma;\xi) = \mathbb{E}\left[e^{-2\pi i\langle\sigma,\xi\rangle}\right], \quad \chi(\sigma^\infty;\xi) = \mathbb{E}\left[e^{-2\pi i\langle\sigma^\infty,\xi\rangle}\right].$$

Since $\langle\sigma_\infty, \xi\rangle \equiv \langle\sigma, \xi\rangle$ mod 1 almost surely,

$$\chi(\sigma;\xi) = \chi(\sigma^\infty;\xi). \tag{9}$$

Now, since σ is almost surely stabilizable, Lemma 1.1 allows us to write $\mathbb{E}[\sigma_0(x)] = \mathbb{E}[\sigma_\infty(x)] = 3 - \epsilon$, for some $\epsilon \geq 0$.

Noting that $\sum_{x\in\mathbb{Z}^2} \xi(x) = 0$, we have $-2\pi i\langle -3, \xi\rangle = 0$. Then by bilinearity of the pairing $\langle\cdot,\cdot\rangle$,

$$\begin{aligned}\exp\left[-2\pi i\langle\sigma_\infty - 3, \xi\rangle\right] &= \exp\left[-2\pi i\langle\sigma_\infty, \xi\rangle\right] \cdot \exp\left[-2\pi i\langle -3, \xi\rangle\right] \\ &= \exp\left[-2\pi i\langle\sigma_\infty, \xi\rangle\right].\end{aligned} \tag{10}$$

So, using (10) in the third line, Lemma 3.5 in the fifth line, and finally monotone convergence for non-negative sums in the seventh line,

$$
\begin{aligned}
1-|\chi(\sigma^\infty;\xi)| &\le |1-\chi(\sigma^\infty;\xi)| \\
&= |\mathbb{E}\left[1-e^{-2\pi i\langle\sigma^\infty,\xi\rangle}\right]| \\
&= |\mathbb{E}\left[1-e^{-2\pi i\langle\sigma^\infty-3,\xi\rangle}\right]| \\
&\le \mathbb{E}\left[|1-e^{-2\pi i\langle\sigma^\infty-3,\xi\rangle}|\right] \\
&\le \mathbb{E}\left[2\pi|\langle\sigma^\infty-3,\xi\rangle|\right] \\
&\le 2\pi\cdot\mathbb{E}\left[\sum_{x\in\mathbb{Z}^2}|[\sigma_\infty(x)-3]\cdot\xi(x)|\right] \\
&\overset{\text{MON}}{=} 2\pi\cdot\sum_{x\in\mathbb{Z}^2}|\xi(x)|\cdot\mathbb{E}\left[|\sigma_\infty(x)-3|\right] \\
&= 2\pi\cdot||\xi||_1\cdot\epsilon.
\end{aligned}
$$

Thus, we may conclude,

$$
|\chi(\sigma^\infty;\xi)| \ge 1-2\pi||\xi||_1\epsilon. \tag{11}
$$

Furthermore, since $(\sigma(x))_{x\in\mathbb{Z}^2}$ are i.i.d random variables,

$$
\begin{aligned}
\chi(\sigma;\xi) &= \mathbb{E}\left[\exp\{\sum_{x\in\mathbb{Z}^2}-2\pi i\sigma(x)\xi(x)\}\right] \\
&= \mathbb{E}\left[\prod_{x\in\mathbb{Z}^2}e^{-2\pi i\sigma(x)\xi(x)}\right] \\
&= \prod_{x\in\mathbb{Z}^2}\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right].
\end{aligned} \tag{12}
$$

Now, using (12) as well as the easily verifiable inequality $-\log(t)\ge\frac{1-t^2}{2}$, for $0<t\le 1$, we obtain,

$$
\begin{aligned}
-\log|\chi(\sigma;\xi)| &= -\log\prod_{x\in\mathbb{Z}^2}|\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]| \\
&= \sum_{x\in\mathbb{Z}^2}-\log|\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]| \\
&\ge \frac{1}{2}\sum_{x\in\mathbb{Z}^2}\left(1-|\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]|^2\right)
\end{aligned} \tag{13}
$$

Application of the inequality is justified since for all $x\in\mathbb{Z}^2$,

$$
0<|\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]| \le \mathbb{E}\left[|e^{-2\pi i\sigma(0)\xi(x)}|\right]=1.
$$

Let X, X' be independent and distributed as $\sigma(0)$, then for all $x \in \mathbb{Z}^2$,

$$
\begin{aligned}
|\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]|^2 &= \overline{\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right]}\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right] \\
&= \mathbb{E}\left[e^{2\pi i\sigma(0)\xi(x)}\right]\mathbb{E}\left[e^{-2\pi i\sigma(0)\xi(x)}\right] \\
&= \mathbb{E}\left[e^{2\pi i X'\xi(x)}\right]\mathbb{E}\left[e^{-2\pi i X\xi(x)}\right] \\
&= \mathbb{E}\left[e^{-2\pi i(X-X')\xi(x)}\right] \\
&= \mathbb{E}\left[\mathrm{Re}[e^{-2\pi i(X-X')\xi(x)}]\right] \\
&= \mathbb{E}\left[\cos\{2\pi(X-X')\xi(x)\}\right] \qquad (14)
\end{aligned}
$$

Where we used that for any random variable Z, $\overline{\mathbb{E}[Z]} = \mathbb{E}[\overline{Z}]$ and $Re(\mathbb{E}[Z]) = \mathbb{E}[Re(Z)]$, both of which may easily be derived from the definition of complex RV's. Now, combining (13) and (14),

$$
-\log|\chi(\sigma;\xi)| \geq \frac{1}{2}\sum_{x\in\mathbb{Z}^2}\left(1-\mathbb{E}\left[\cos\{2\pi(X-X')\xi(x)\}\right]\right)
$$

Further, since each term in the sum is non-negative, we can restrict the sum to the set $A = \{x \in \mathbb{Z}^2 : 0 < |(X-X')\xi(x)| \leq \frac{1}{2}\}$ and apply the inequality $1 - cos(2\pi t) \geq 8t^2$ for $|t| \leq \frac{1}{2}$,

$$
\begin{aligned}
-\log|\chi(\sigma;\xi)| &\geq \frac{1}{2}\sum_{x\in A}\mathbb{E}\left[1-\cos\{2\pi(X-X')\xi(x)\}\right] \\
&\geq 4\sum_{x\in A}\mathbb{E}\left[(X-X')^2\xi(x)^2\right] \\
&\overset{\text{MON}}{=} 4\mathbb{E}\left[\sum_{x\in A}(X-X')^2\xi(x)^2\right] \\
&= 4\mathbb{E}\left[\sum_{k=1}^{\infty}\mathbb{1}[|X-X'|=k]\sum_{0<|\xi(x)|k\leq\frac{1}{2}}k^2\xi(x)^2\right] \\
&\overset{\text{MON}}{=} 4\sum_{k=1}^{\infty}\mathbb{E}\left[\mathbb{1}[|X-X'|=k]\sum_{0<|\xi(x)|\leq\frac{1}{2k}}k^2\xi(x)^2\right] \qquad (15)
\end{aligned}
$$

Where we used monotone convergence for non-negative sums as well as the law of total expectation. Now, by Lemma 3.6 combined with (15), we deduce

$$\begin{aligned}
-\log|\chi(\sigma;\xi)| &\geq 4\sum_{k=1}^{\infty}\left[\sum_{0<|\xi(x)|\leq\frac{1}{2k}} k^2\xi(x)^2\right]\mathbb{E}\left[\mathbb{1}|X-X'|=k\right] \\
&\gg \sum_{k=1}^{\infty}\left[k^2k^{-\frac{4}{3}}\right]\mathbb{E}\left[\mathbb{1}|X-X'|=k\right] \\
&= \sum_{k=1}^{\infty} k^{\frac{2}{3}}\mathbb{E}\left[\mathbb{1}|X-X'|=k\right] \\
&= \mathbb{E}\left[|X-X'|^{\frac{2}{3}}\right].
\end{aligned}$$

Consequently, splitting into 2 cases we see

- If $|\chi(\sigma;\xi)| \in [\frac{1}{2}, 1]$, then

$$\begin{aligned}
&1-|\chi(\sigma;\xi)| \geq -\frac{1}{2}\log|\chi(\sigma;\xi)| \\
\implies\ &1-|\chi(\sigma;\xi)| \gg -\log(|\chi(\sigma;\xi)|) \gg \mathbb{E}\left[|X-X'|^{\frac{2}{3}}\right].
\end{aligned}$$

- If $|\chi(\sigma;\xi)| \in [0, \frac{1}{2}]$, then $1-|\chi(\sigma;\xi)| \gg 1$.

Thus, we conclude

$$1-|\chi(\sigma^{\infty};\xi)| \gg \min(1, \mathbb{E}[|X-X'|^{\frac{2}{3}}]).$$

□

4 Analogue to Theorem 1 on $\mathbb{Z} \times \mathbb{Z}_k$

We now investigate the behaviour of the sandpile on the graph $\mathbb{Z}\times\mathbb{Z}_k$ and derive a statement analogous to Theorem 1.

4.1 Preliminaries

First, we recall the notion of a Poisson Process.

Definition 4.1 (Poisson Process)**.** *The homogeneous Poisson process, when considered on the positive half-line, can be defined as a counting process, denoted as $\{N(t), t \geq 0\}$. A counting process is a homogeneous Poisson counting process with rate $\lambda > 0$ if it has the following properties:*

- $N(0) = 0$*;*
- *the process has independent increments;*
- *the number of points in any interval of length t is a Poisson RV with parameter λt.*

For a general Poisson process $N(t)$, let $T_0 = 0$ and define for all $n \in \mathbb{N}$,

$$T_n = \inf\{t : N(t) = n\},$$
$$W_n = T_n - T_{n-1},$$

to be the $n'th$ arrival time and $n'th$ interarrival time, respectively.

Note that we may refer to the Poisson process as a 'Poisson clock' in circumstances where we only care about the interarrival times. This distinction makes it clear we're not concerened with the value of the Poisson process at any given time, just the duration of time between arrivals.

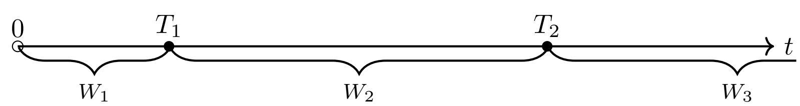

Figure 2: *A Poisson process on the positive half line.*

Lemma 4.1. *Arrival times of the Poisson process have the 'memoryless' property. This means if $N(t)$ is a Poisson process of rate λ and $t, s \geq 0$,*

$$\mathbb{P}[W_n > t + s | W_n > t] = \mathbb{P}[W_n > s].$$

Proof. Noting $(W_n)_{n\geq 1}$ are i.i.d random variables due to the 'independent increments' of the Poisson process, showing $W_1 \sim \text{Exp}(\lambda)$ will suffice since the exponential distribution posseses the memoryless property. Fix $t > 0$,

$$\begin{aligned}\mathbb{P}[W_1 \leq t] &= \mathbb{P}[T_1 \leq t] \\ &= \mathbb{P}[N(t) \neq 0] \\ &= 1 - e^{-\lambda t}\end{aligned}$$

using that for fixed $t > 0$ we have $N(t) \sim \text{Po}(\lambda t)$. This is the CDF of an exponential distribution with parameter λ, so the claim holds. □

Equivalently, the Poisson process is in fact charatcerised by having these Exponentially distributed interarrival times.

Lemma 4.2. *Suppose $N(t)$ is a counting process with $N(0) = 0$. Then N is a Poisson process with rate λ if and only if*

- *interarrival times are i.i.d. Exp(λ) random variables;*
- *the 'jump chain' is simply the enumeration of the natural numbers.*

Proof. For details one may consult [4]. The equivalence of these definitions is commonplace and proofs can be found in many other textbooks also. □

For completeness we prove an elementary result which will be useful later.

Lemma 4.3. *For a constant c such that $|c| < 1$, the sum $\sum_{x=0}^{\infty} xc^x$ is covergent.*

Proof. Taking the derivative of the following expression

$$\sum_{x=0}^{\infty} c^x = \frac{1}{1-c}$$

we obtain

$$\sum_{x=1}^{\infty} xc^{x-1} = \frac{1}{(1-c)^2}.$$

This implies,

$$\sum_{x=0}^{\infty} xc^x = \left[\sum_{x=0}^{\infty}(x+1)c^x\right] - \left[\sum_{x=0}^{\infty} c^x\right] = \frac{1}{(1-c)^2} - \frac{1}{1-c} = \frac{c}{(1-c)^2} < \infty$$

□

4.2 Greens Function on $\mathbb{Z} \times \mathbb{Z}_k$

Let p_n, p'_n be the transition probabilities corresponding to the following discrete-time random walks S_n, S'_n respectively,

- $(S_n)_{n\geq 0}$ is the simple symmetric random walk on $\mathbb{Z} \times \mathbb{Z}_k$,
- $(S'_n)_{n\geq 0}$ is the random walk on $\mathbb{Z}\times\mathbb{Z}_k$ which doesn't move with probability $\frac{1}{2}$ and otherwise moves to adjacent verticies with equal probability.

Definition 4.2. *Define the following function on $\mathbb{Z} \times \mathbb{Z}_k$,*

$$G_{\mathbb{Z}\times\mathbb{Z}_k}(x) = \frac{1}{2} \lim_{N\to\infty} \sum_{n=0}^{N} p'_n(x) - p'_n(0).$$

Remark Denoting this function 'G' is consistent with our definition of Greens function in the case of $\mathbb{Z}^2$, since it can be shown that for each $x \in \mathbb{Z} \times \mathbb{Z}_k$

$$\frac{1}{2} \lim_{N\to\infty} \sum_{n=0}^{N} p'_n(x) - p'_n(0) = \frac{1}{4} \lim_{N\to\infty} \sum_{n=0}^{N} p_n(x) - p_n(0)$$

where the right hand side is how we'd expect to define $G_{\mathbb{Z}\times\mathbb{Z}_k}$. Since we don't require this fact however, the proof of the claim is omitted.

Definition 4.3. *Consider a general discrete-time random walk $(S_n)_{n\geq 0}$ with distribution q_n. Let $\{N(t) : t \geq 0\}$ be an independent Poisson process with parameter $\lambda = 1$.*

Then, define the corresponding continuous time random walk $(\tilde{S}_t)_{t\geq 0}$ by

$$\tilde{S}_t = S_{N(t)}, \text{ for } t \geq 0.$$

Finally, for $t \geq 0$, call $\tilde{q}_t(x) = \mathbb{P}[\tilde{S}_t = x]$ the corresponding continuous time transition probabilities.

The continuous-time random walk $\tilde{S}_t$ follows the same path as the discrete time random walk S_n, however instead of making jumps at discrete time intervals, the process moves at times corresponding to the Poisson process $N(t)$.

Proposition 4.1. *$G_{\mathbb{Z}\times\mathbb{Z}_k}(x)$ is well-defined and can be calculated as*

$$G_{\mathbb{Z}\times\mathbb{Z}_k}(x) = \int_0^\infty \tilde{p}_t(x) - \tilde{p}_t(0)\, dt.$$

where $\tilde{p}_t$ are the continuous time transition probabilities corresponding to p_n.

Proof. Notice that $\tilde{p}$ is equivalently defined as

$$\tilde{p}_t(x) = \sum_{n=0}^{\infty} e^{-2t} \frac{(2t)^n}{n!} p'_n(x).$$

This can be seen from the fact that it's equally likely to jump in all directions and interarrival times are i.i.d Exp(1) RV's (due to the fact that summing Geom($\frac{1}{2}$) independent Exp(2) RV's yields a Exp(1) RV), thus the claim follows by Lemma 4.2.

We proceed by first showing absolute convergence of the sum defining $G_{\mathbb{Z}\times\mathbb{Z}_k}$, i.e. for all $x \in \mathbb{Z}\times\mathbb{Z}_k$,

$$\sum_{n=0}^{\infty}|p'_n(x) - p'_n(0)| < \infty. \tag{16}$$

To prove this claim we must also define p''_n to be the transition probabilities of a RW on $\mathbb{Z}^2$ analogous to p'_n, i.e. not moving with probability $\frac{1}{2}$ and moving to adjacent verticies with equal probability otherwise.

Then, crucially, by 'wrapping $\mathbb{Z}^2$ around $\mathbb{Z}\times\mathbb{Z}_k$' we notice that for all $x \in \mathbb{Z}^2$

$$p'_n(x) = \sum_{m\in\mathbb{Z}} p''_n(x + mke_2)$$

where $e_2 = (0, 1)$. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}|p'_n(x) - p'_n(0)| &= \sum_{n=0}^{\infty}\Big|\sum_{m\in\mathbb{Z}} p''_n(x + mke_2) - p''_n(mke_2)\Big| \\ &\le \sum_{n=0}^{\infty}|p''_n(x) - p''_n(0)| \qquad (17) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}|p''_n(x + mke_2) + p''_n(x - mke_2) - 2p''_n(mke_2)|.\end{aligned}$$

Since p'' is an aperiodic walk on $\mathbb{Z}^2$, the sum on line (17) converges as proved in [7] (Section 4.4.1), specifically this is because

$$|p''_n(x) - p''_n(0)| \le c|x|n^{-\frac{3}{2}}.$$

To deal with the double sum, write $\bar{p}_n(z)$ for Gaussian density with covariance

$$\Gamma = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

which is since $\mathbb{E}[X_1^2] = (-1)^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{1}{8} = \frac{1}{4} = \mathbb{E}[X_2^2]$ and $\mathbb{E}[X_1X_2] = 0$, where X_1, X_2 correspond to the vertical and horizontal component of p'', then

$$\bar{p}_n(z) = \frac{C}{n}\exp\left(-\frac{z\Gamma^{-1}z}{2n}\right) = \frac{C}{n}\exp\left(-\frac{2|z|^2}{n}\right).$$

For a generic random walk p (on $\mathbb{Z}^2$) define ∇ such that for $x, y \in \mathbb{Z}^2$,

$$\nabla_x p(y) = p(x+y) - p(y).$$

Then noting that $p''(\cdot)$ is an even function by symmetry of the random walk,

$$\begin{aligned}|p''_n(x+mke_2) + p''_n(x-mke_2) - 2p''_n(mke_2)| &= |\nabla_x p''_n(mke_2) + \nabla_{-x} p''_n(mke_2)| \\ &\le |\nabla_x \bar{p}_n(mke_2) + \nabla_{-x}\bar{p}_n(mke_2)| \\ &+ O(n^{-\frac{5}{2}})\end{aligned}$$

where the inequality is by Theorem 2.3.6 of [7]. Then sum over m such that $|mke_2| \le n^{\frac{1}{2}+\delta}$ for some small $\delta > 0$, noticing

$$\sum_{m:|mk|\le n^{\frac{1}{2}+\delta}} O(n^{-\frac{5}{2}}) = O(n^{-2+\delta})$$

which is summable in n. Further, calculating by definition of ∇ and $\bar{p}_n$,

$$\begin{aligned}|\nabla_x \bar{p}_n(mke_2) + \nabla_{-x}\bar{p}_n(mke_2)| =& \frac{C}{n}|\exp\left(-\frac{2|mke_2+x|^2}{n}\right) \\ &+ \exp\left(-\frac{2|mke_2-x|^2}{n}\right) - 2\exp\left(-\frac{2|mke_2|^2}{n}\right)| \\ =& \frac{C}{n}\exp\left(-\frac{2(mk)^2}{n}\right)|\exp\left(-\frac{2mk}{n}\langle e_2, x\rangle\right) \\ &+ \exp\left(-\frac{2mk}{n}\langle e_2, -x\rangle\right) + O(n^{-1}) - 2| \\ =& \frac{C}{n}\exp\left(-\frac{2(mk)^2}{n}\right)|1 - \frac{2mk}{n}\langle e_2, x\rangle \\ &+ 1 - \frac{2mk}{n}\langle e_2, -x\rangle + O(n^{-1}) - 2| \\ =& \frac{C}{n}\exp\left(-\frac{2(mk)^2}{n}\right)\cdot O(n^{-1}) \\ \le& \frac{\tilde{C}}{n^2}.\end{aligned}$$

Thus

$$\sum_{m:|mk|\le n^{\frac{1}{2}+\delta}} |\nabla_x \bar{p}_n(mke_2) + \nabla_{-x}\bar{p}_n(mke_2)| \le \sum_{m:|mk|\le n^{\frac{1}{2}+\delta}} \frac{\tilde{C}}{n^2} = \frac{\overline{C}}{n^{\frac{3}{2}-\delta}}$$

which is summable in n. We then still want to bound the term,

$$\sum_{n=0}^{\infty} \sum_{m:|mk|>n^{\frac{1}{2}+\delta}} |p''_n(x+mke_2) + p''_n(x-mke_2) - 2p''_n(mke_2)|,$$

to this end, notice that for S''_n the random walk corresponding to p''_n,

$$\sum_{m:|mk|>n^{\frac{1}{2}+\delta}} |p''_n(x+mke_2)+p''_n(x-mke_2)-2p''_n(mke_2)| \le \mathbb{P}[|S''_n| \ge n^{\frac{1}{2}+\frac{\delta}{2}}].$$

Additionally, by Corollary 12.2.7 of [7], using $r = n^{\frac{\delta}{2}}$,

$$\begin{aligned}\mathbb{P}[|S''_n| \ge n^{\frac{1}{2}+\frac{\delta}{2}}] &\le \exp(-cn^{\delta}) \cdot \exp\left(O\left(\frac{n^{\frac{3\delta}{2}}}{n^{\frac{1}{2}}}\right)\right) \\ &\le C\exp(-cn^{\delta})\end{aligned}$$

noting that $\frac{3\delta}{2} - \frac{1}{2} < 0$ for $\delta > 0$ small. Then this is also summable in n, which completes the proof of statement (16), i.e. $G_{\mathbb{Z}\times\mathbb{Z}_k}$ is well-defined and moreover the sum converges absolutely.

We now also claim that

$$\int_0^\infty |\tilde{p}_t(x) - \tilde{p}_t(0)|\, dt < \infty, \tag{18}$$

$$\int_0^\infty \tilde{p}_t(x) - \tilde{p}_t(0)\, dt = \frac{1}{2}\sum_{n=0}^{\infty}(p'_n(x) - p'_n(0)). \tag{19}$$

First we prove (18) by calculating the following, using Tonelli's theorem in the second line since the terms are non-negative,

$$\begin{aligned}\int_0^\infty |\tilde{p}_t(x) - \tilde{p}_t(0)|\, dt &\le \int_0^\infty \sum_{n=0}^{\infty} e^{-2t}\frac{(2t)^n}{n!}|p'_n(x) - p'_n(0)| \\ &= \sum_{n=0}^{\infty}|p'_n(x) - p'_n(0)| \int_0^\infty e^{-2t}\frac{(2t)^n}{n!}\, dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty}|p'_n(x) - p'_n(0)| \\ &< \infty\end{aligned}$$

where we proved the last sum is finite by (16). Then using Fubinis theorem we can do the same again without the absolute values to obtain (19). □

4.3 Bounding $D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l)$ from below.

We must now provide a lower bound for $D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l)$ to be used in place of Lemma 3.6.

Theorem 4.1. *For some constant $C > 0$ which depends on k, we have*

$$D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l) \gg e^{-C|x|}.$$

Proof. By Proposition 4.1, $D_2G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l)$ may be written as

$$\begin{aligned} D_2(x,l) &= \int_0^\infty \tilde{p}_t(x,l+1) - \tilde{p}_t(x,l)\; dt \\ &= \int_0^\infty \mathbb{P}[\tilde{S}_t^1 = x]\cdot\Big[\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]\Big]\; dt \end{aligned}$$

We proceed by splitting up the proof into 5 smaller steps.

Step 1: Calculating $\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]$.

Noting that the random walk moves in each vertical direction with rate $\frac{1}{4}$ to ensure it leaves states with rate 1, the Q-matrix of $\tilde{S}^2$ is given by

$$Q = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & 0 & \cdots & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \cdots & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & 0 & 0 & \cdots & -\frac{1}{2} \end{pmatrix}$$

Then, since Q is symmetric, it is diagonalisable. For each $j = 0, ..., k-1$, define the function

$$f_j(m) = \exp\left(2\pi i\cdot\frac{mj}{k}\right), \text{ for } m \in \mathbb{Z}_k.$$

We wish to show the functions $f_0, ..., f_{k-1}$ give rise to the eigenvectors of the diagonalisable Q and thus give rise to an eigenbasis. To that end,

$$\begin{aligned} (Qf_j)(m) &= -\frac{1}{2}f_j(m) + \frac{1}{4}f_j(m-1) + \frac{1}{4}f_j(m+1) \\ &= -\frac{1}{4}\left(2e^{2\pi i\frac{mj}{k}} - e^{2\pi i\frac{(m-1)j}{k}} - e^{2\pi i\frac{(m+1)j}{k}}\right) \\ &= -\frac{1}{4}e^{2\pi i\frac{mj}{k}}\left(2 - e^{-2\pi i\frac{j}{k}} - e^{2\pi i\frac{j}{k}}\right) \\ &= -\frac{1}{4}e^{2\pi i\frac{mj}{k}}\left(2 - 2\cos\left(2\pi\cdot\frac{j}{k}\right)\right) \\ &= f_j(m)\left(\frac{\cos\left(2\pi\cdot\frac{j}{k}\right) - 1}{2}\right) \end{aligned}$$

implying for $j = 0, ..., k-1$ each f_j indeed gives an eigenvector with corresponding eigenvalue

$$\lambda_j = \frac{\cos\left(2\pi\cdot\frac{j}{k}\right) - 1}{2} = -\sin^2\left(\frac{\pi j}{k}\right).$$

Thus, by a standard linear algebra result, Q may be written as $Q = SDS^{-1}$ where S consists of eigenvectors on the columns and D is the diagonal matrix of eigenvalues. Explicitly, noting that $\lambda_0 = 0$,

$$S = \begin{pmatrix} 1 & f_1(0) & \cdots & f_{k-1}(0) \\ 1 & f_1(1) & \cdots & f_{k-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(k-1) & \cdots & f_{k-1}(k-1) \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1} \end{pmatrix}$$

Then

$$\begin{aligned} e^{tQ} &= \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!} \\ &= S\left[\sum_{n=0}^{\infty} \frac{(tD)^n}{n!}\right] S^{-1} \\ &= S\mathrm{diag}(1, e^{t\lambda_1}, ..., e^{t\lambda_{k-1}})S^{-1} \\ &= \sum_{j=0}^{k-1} e^{t\lambda_j} S E_j S^{-1} \\ &= \sum_{j=0}^{k-1} e^{t\lambda_j} U_j \end{aligned}$$

where for $j = 0, ..., k-1$, E_j=diag$(0, ..., 1, ..., 0)$ with the 1 on the j'th diagonal, and $U_j = SE_jS^{-1}$. Since S is unitary (up to scalar), it's inverse is it's conjugate transpose multiplied by $\frac{1}{k}$, i.e.

$$S^{-1} = \frac{1}{k}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \overline{f_1(0)} & \overline{f_1(1)} & \cdots & \overline{f_1(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{f_{k-1}(0)} & \overline{f_{k-1}(1)} & \cdots & \overline{f_{k-1}(k-1)} \end{pmatrix}$$

so we calculate U_j as follows,

$$U_j = \frac{1}{k}\begin{pmatrix} 1 & f_1(0) & \cdots & f_{k-1}(0) \\ 1 & f_1(1) & \cdots & f_{k-1}(1) \\ 1 & f_1(2) & \cdots & f_{k-1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(k-1) & \cdots & f_{k-1}(k-1) \end{pmatrix}\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \overline{f_j(0)} & \cdots & \cdots & \overline{f_j(k-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} f_j(0)\overline{f_j(0)} & \cdots & f_j(0)\overline{f_j(k-1)} \\ f_j(1)\overline{f_j(0)} & \cdots & f_j(1)\overline{f_j(k-1)} \\ \vdots & \ddots & \vdots \\ f_j(k-1)\overline{f_j(0)} & \cdots & f_j(k-1)\overline{f_j(k-1)} \end{pmatrix}.$$

Then, noting $\lambda_1 = \lambda_{k-1} = -\sin^2(\frac{\pi}{k})$,

$$\begin{aligned} \mathbb{P}_0[\tilde{S}_t = l] &= (e^{tQ})_{0l} \\ &= \frac{1}{k} + e^{\lambda_1 t}\left[(U_1)_{0l} + (U_{k-1})_{0l}\right] + \sum_{j=2}^{k-2} e^{\lambda_j t}(U_j)_{0l}. \end{aligned} \tag{20}$$

So we proceed by calculating

$$(U_1)_{0l} = \frac{1}{k} f_1(0)\overline{f_1(l)} = \frac{1}{k}\overline{f_1(l)} = \frac{1}{k}e^{-2\pi i \frac{l}{k}}$$

$$(U_{k-1})_{0l} = \frac{1}{k} f_{k-1}(0)\overline{f_{k-1}(l)} = \frac{1}{k}\overline{f_{k-1}(l)} = \frac{1}{k}e^{2\pi i \frac{l}{k}}$$

$$\implies (U_1)_{0l} + (U_{k-1})_{0l} = \frac{2}{k}\cos(2\pi\frac{l}{k}). \tag{21}$$

Similairly, we can derive that

$$(U_1)_{0,l+1} + (U_{k-1})_{0,l+1} = \frac{2}{k}\cos(2\pi\frac{l+1}{k}). \tag{22}$$

Putting together (19), (20), (21) yields

$$\begin{aligned} \mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l] = \frac{2}{k}e^{\lambda_1 t}&\left[\cos(2\pi\frac{l+1}{k}) - \cos(2\pi\frac{l}{k})\right] \\ &+ \sum_{j=2}^{k-2} e^{\lambda_j t}\left[(U_j)_{0,l+1} - (U_j)_{0,l}\right]. \end{aligned}$$

Thus we calculate

$$\begin{aligned}
\cos(2\pi\frac{l+1}{k}) - \cos(2\pi\frac{l}{k}) &= \mathrm{Re}\left(e^{2\pi i\frac{l+1}{k}} - e^{2\pi i\frac{l}{k}}\right) \\
&= \mathrm{Re}\left(e^{2\pi i\frac{l+\frac{1}{2}}{k}}\left[e^{2\pi i\frac{1}{2k}} - e^{-2\pi i\frac{1}{2k}}\right]\right) \\
&= \mathrm{Re}\left(\left[\cos(...) + i\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\right]2i\sin\left(\frac{\pi}{k}\right)\right) \\
&= -2\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\sin\left(\frac{\pi}{k}\right).
\end{aligned}$$

Combining yields the equality

$$\begin{aligned}
\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l] = &-\frac{4}{k}e^{\lambda_1 t}\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\sin\left(\frac{\pi}{k}\right) \\
&+ \sum_{j=2}^{k-2} e^{\lambda_j t}\left[(U_j)_{0,l+1} - (U_j)_{0,l}\right].
\end{aligned}$$

Step 2: Bounding $|\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]|$ from below (for large t).

We first claim that for $j = 2, ..., k-2$ that $e^{\lambda_j t} = o(e^{\lambda_1 t})$. To see this we calculate

$$e^{\lambda_j t} \cdot e^{-\lambda_1 t} = e^{(\lambda_j - \lambda_1)t} \to 0, \text{ as } t \to \infty,$$

since for each such j, $\lambda_j - \lambda_1 < 0$. From this we can deduce that for each l,

$$\sum_{j=2}^{k-2} e^{\lambda_j t}\left[|(U_j)_{0,l+1}| + |(U_j)_{0,l}|\right] = o\left(e^{\lambda_1 t}\left|\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\right|\sin\left(\frac{\pi}{k}\right)\right),$$

thus we can choose t_0 such that for all $t > t_0$ and all l,

$$\sum_{j=2}^{k-2} e^{\lambda_j t}\left[|(U_j)_{0,l+1}| + |(U_j)_{0,l}|\right] < \frac{1}{k}e^{\lambda_1 t}\left|\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\right|\sin\left(\frac{\pi}{k}\right). \tag{23}$$

Then, using the triangle inequality and (23) in the last line, for all $t > t_0$ and all l,

$$
\begin{aligned}
|\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]| &\geq |-\frac{4}{k}e^{\lambda_1 t}\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)\sin\left(\frac{\pi}{k}\right)| \\
&\quad - |\sum_{j=2}^{k-2} e^{\lambda_j t}\left[(U_j)_{0,l+1} - (U_j)_{0,l}\right]| \\
&\geq \frac{4}{k}e^{\lambda_1 t}|\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)|\sin\left(\frac{\pi}{k}\right) \\
&\quad - \sum_{j=2}^{k-2} e^{\lambda_j t}\left[|(U_j)_{0,l+1}| + |(U_j)_{0,l}|\right] \\
&> c_1 e^{\lambda_1 t}
\end{aligned}
$$

where $c_1 = \frac{3}{k}|\sin\left(2\pi\frac{l+\frac{1}{2}}{k}\right)|\sin\left(\frac{\pi}{k}\right)$. As a result of this we can also conclude that $\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]$ does not change sign for all $t > t_0$.

Step 3: Bounding $\mathbb{P}[\tilde{S}_t^1 = x]$ from below.

Firstly, note that $\tilde{S}^1$ is the continuous time random walk corresponding to the simple symmetric random walk on $\mathbb{Z}$, except that the corresponding Poisson process has rate $\frac{1}{2}$ as opposed to the standard rate 1. With this in mind we can apply Theorem 2.5.6 of [7] to obtain that if $|x| < \frac{t}{4}$

$$
\mathbb{P}[\tilde{S}_t^1 = x] = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{x^2}{t}\right)\exp\left(O\left(\frac{1}{\sqrt{t}} + \frac{|x|^3}{t^2}\right)\right).
$$

Let c_2 be a constant such that the error term in the exponential is less than $c_2\left(\frac{1}{\sqrt{t}} + \frac{|x|^3}{t^2}\right)$. Choose $c_3 \geq 4$ such that for $|t| \geq c_3|x|$,

$$
c_2\left(\frac{1}{\sqrt{t}} + \frac{|x|^3}{t^2}\right) \;\leq\; c_2\left(\frac{1}{\sqrt{t}} + \frac{t}{c_3^3}\right) \;\leq\; \frac{2c_2}{c_3^3}t \;\leq\; \frac{1}{c_3^2}t.
$$

Then for $t \geq c_3|x|$ we have

$$
\mathbb{P}[\tilde{S}_t^1 = x] \geq \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{x^2}{t}\right)\exp\left(-\frac{t}{c_3^2}\right) \geq \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{2}{c_3^2}t\right).
$$

Step 4: Bounding $\int_0^{t_0}\mathbb{P}[\tilde{S}_t^1 = x]$ from above.

Let $\{N^1(t) : t \geq 0\}$ be the Poisson process of rate $\frac{1}{2}$ determining the jump times of $\tilde{S}^1$. Then noting that the Poisson clock must ring at least $|x|$ times in the interval $[0,t]$ for the event $\{\tilde{S}_t^1 = x\}$ to have positive probability, we have for any $\theta > 0$,

$$
\begin{aligned}
\int_0^{t_0} \mathbb{P}[\tilde{S}_t^1 = x]\, dt &\leq \int_0^{t_0} \mathbb{P}[N^1(t) \geq |x|]\, dt \\
&\leq t_0 \cdot \mathbb{P}[N^1(t_0) \geq |x|] \\
&= t_0 \cdot \mathbb{P}[e^{\theta N^1(t_0)} \geq e^{\theta |x|}] \\
&\leq t_0 \cdot \frac{\mathbb{E}[e^{\theta N^1(t_0)}]}{e^{\theta|x|}} \\
&= t_0 e^{-\theta|x|} \sum_{n=0}^{\infty} e^{\theta n} \cdot \mathbb{P}[N^1(t_0) = n] \\
&= t_0 e^{-\theta|x|} \sum_{n=0}^{\infty} e^{\theta n} \cdot e^{-\frac{1}{2}t_0} \frac{(\frac{1}{2}t_0)^n}{n!} \\
&= t_0 e^{-\theta|x| - \frac{1}{2}t_0} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}t_0 \cdot e^{\theta})^n}{n!} \\
&= t_0 \exp\left(-\theta|x| + \frac{1}{2}t_0(e^{\theta} - 1)\right) \\
&\leq t_0 \exp\left(-\theta|x| + \frac{1}{2}t_0 e^{\theta}\right).
\end{aligned}
$$

So in particular if we choose $\theta = \log|x|$, we have for all $x \in \mathbb{Z}$ such that $|x| > e^{2t_0}$,

$$
\begin{aligned}
\int_0^{t_0} \mathbb{P}[\tilde{S}_t^1 = x]\, dt &\leq t_0 \exp\left(-|x|\log|x| + \frac{1}{2}|x|t_0\right) \\
&= t_0 \exp\left(-|x|\left(\log|x| - \frac{1}{2}t_0\right)\right) \\
&\leq t_0 \exp\left(-\frac{3}{4}|x|\log|x|\right).
\end{aligned}
$$

Step 5: Completing the proof.

Recalling that we are trying to lower bound

$$
D_2 G_{\mathbb{Z}\times\mathbb{Z}_k} = \int_0^{\infty} \mathbb{P}[\tilde{S}_t^1 = x] \cdot \left[\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]\right]\, dt,
$$

we use the previous steps to proceed as follows

$$
\begin{aligned}
D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l) &\geq \int_{t_0}^{\infty} \mathbb{P}[\tilde{S}_t^1 = x]\cdot \Big[\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]\Big]\ dt \\
&\quad - \int_0^{t_0} \mathbb{P}[\tilde{S}_t^1 = x]\cdot \Big[\mathbb{P}[\tilde{S}_t^2 = l+1] - \mathbb{P}[\tilde{S}_t^2 = l]\Big]\ dt \\
&\geq \int_{t_0}^{\infty} \mathbb{P}[\tilde{S}_t^1 = x] c_1 e^{\lambda_1 t}\ dt - \int_0^{t_0} \mathbb{P}[\tilde{S}_t^1 = x]\ dt \\
&\geq \int_0^{\infty} \mathbb{P}[\tilde{S}_t^1 = x] c_1 e^{\lambda_1 t}\ dt - (1+c_1)t_0 \exp\left(-\frac{3}{4}|x|\log|x|\right).
\end{aligned}
$$

However for $U \sim \mathrm{Exp}(\lambda_1)$ independent of $\tilde{S}_t^1$,

$$
\begin{aligned}
\int_0^{\infty} \mathbb{P}[\tilde{S}_t^1 = x] c_1 e^{\lambda_1 t}\ dt &= \frac{c_1}{\lambda_1}\int_0^{\infty} \mathbb{P}[\tilde{S}_t^1 = x]\lambda_1 e^{\lambda_1 t}\ dt \\
&= \frac{c_1}{\lambda_1}\mathbb{P}[\tilde{S}_U^1 = x].
\end{aligned}
$$

Additionally, using step 2 in the second inequality, we calculate

$$
\begin{aligned}
\frac{c_1}{\lambda_1}\mathbb{P}[\tilde{S}_U^1 = x] &\geq \frac{c_1}{\lambda_1}\mathbb{P}\left[U \in [c_3|x|, c_3|x|+1]\right]\cdot \min_{c_3|x|\leq t\leq c_3|x|+1} \mathbb{P}[\tilde{S}_t^1 = x] \\
&\geq \frac{c_1}{\lambda_1}\exp\left(-c_3\lambda_1|x|\right)(1-e^{-1})\cdot \frac{1}{\sqrt{\pi(c_3|x|+1)}}\exp\left(-\frac{2}{c_3^2}c_3|x|\right) \\
&\geq C\exp\left(-\left(c_3\lambda_1 + \frac{2}{c_3}\right)|x|\right).
\end{aligned}
$$

Combining, we deduce that

$$
D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l) \geq C\exp\left(-\left(c_3\lambda_1 + \frac{2}{c_3}\right)|x|\right) - (1+c_1)t_0\exp\left(-\frac{3}{4}|x|\log|x|\right)
$$

Then since $\exp\left(-|x|\log(|x|)\right) = o(\exp(-|x|))$, we conclude

$$
D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,l) \gg \exp\left(-\left(c_3\lambda_1 + \frac{2}{c_3}\right)|x|\right)
$$

$\square$

4.4 A Coupling Process on $\mathbb{Z}\times\mathbb{Z}_k$.

Let N^1, N^2, N^3 be independent Poisson processes, each with rate $\frac{1}{2}$, and let S^1 be a one dimensional, symmetric, simple random walk (SSRW) and S^2, S^3 be i.i.d SSWR on $\mathbb{Z}_k$. Then, as described earlier, define the continuous-time random walks

$$
\tilde{S}_t^i = S_{N(t)}^i,\ \text{for } i = 1,2,3.
$$

Additionally, we define the coupling time T as follows,

$$T = \inf\{t \geq 0 : \tilde{S}_t^2 \equiv \tilde{S}_t^3 - 1\}.$$

Then we also introduce,

$$\tilde{S}_t^4 = \begin{cases} \tilde{S}_t^2, & \text{for } t \leq T \\ \tilde{S}_t^3 - 1, & \text{for } t > T \end{cases}$$

So $\tilde{S}^4$ is identical to $\tilde{S}^2$ until time T when it is 'one jump below' $\tilde{S}^3$, at which point it couples with $\tilde{S}^3$ and stays one jump below thereafter.

Crucially, by definition of the coupling time, $\tilde{S}^4$ isn't forced to make any jumps and is right-continuous at time T. Additionally, at time T we swap from Poisson process N^2 to N^3. Such a process still serves as an identially distributed Poisson clock in its own right, since we're only concered with interarrival times and the memoryless property ensures they are irrespective of activity before time T.

As a result of the reasoning above, $\tilde{S}^4$ is distributed identically to $\tilde{S}^3$, both of which are independent of $\tilde{S}^1$. It's easily shown that summing independent Poisson processes of rates λ and μ yields a Poisson process of rate $(\lambda + \mu)$, as a result the following continuous-time random walks on $\mathbb{Z} \times \mathbb{Z}_k$

$$(\tilde{S}^1, \tilde{S}^3)$$
$$(\tilde{S}^1, \tilde{S}^4)$$

make jumps at times corresponding to a Poisson clock of rate 1, with jumps going exclusively to adjacent verticies with equal probability. Thus, they're realisations of the continuous-time random walk with ditribution $\tilde{p}$, where p is the distribution of the discrete-time SSRW on $\mathbb{Z} \times \mathbb{Z}_k$.

Proposition 4.2. *The function $D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}$ is in ℓ^1.*

Proof. We proceed using the realisations of $\tilde{p}$ defined above.

$$\begin{aligned} D_2 G_{\mathbb{Z}\times\mathbb{Z}_k}(x,y) &= G_{\mathbb{Z}\times\mathbb{Z}_k}(x,y+1) - G_{\mathbb{Z}\times\mathbb{Z}_k}(x,y) \\ &= \int_0^\infty \tilde{p}_t(x,y+1) - \tilde{p}_t(x,y)\ dt \\ &\leq \int_0^\infty |\tilde{p}_t(x,y+1) - \tilde{p}_t(x,y)|\ dt \end{aligned}$$

Now we bound the expression inside the integral using our previously defined realisations of $\tilde{p}$.

$$\begin{aligned} |\tilde{p}_t(x,y+1) - \tilde{p}_t(x,y)| &= |\mathbb{E}\left[\mathbb{1}(\tilde{S}_t^1, \tilde{S}_t^4) = (x,y+1)\right] - \mathbb{E}\left[\mathbb{1}(\tilde{S}_t^1, \tilde{S}_t^3) = (x,y)\right]| \\ &= |\mathbb{E}\left[\mathbb{1}\{\tilde{S}_t^1 = x\}\mathbb{1}\{\tilde{S}_t^4 = y+1\} - \mathbb{1}\{\tilde{S}_t^1 = x\}\mathbb{1}\{\tilde{S}_t^3 = y\}\right]| \\ &\leq \mathbb{E}|\mathbb{1}\{\tilde{S}_t^1 = x\} \cdot (\mathbb{1}\{\tilde{S}_t^4 = y+1\} - \mathbb{1}\{\tilde{S}_t^3 = y\})| \\ &\leq \mathbb{E}|\mathbb{1}\{\tilde{S}_t^1 = x\}\mathbb{1}\{\tilde{S}_t^4 \neq \tilde{S}_t^3 - 1\}| \end{aligned}$$

Where we use that for any $y \in \mathbb{Z}_k$ the following inequality holds pointwise. To see this, notice both sides are always 1 or 0, and the right hand side equalling zero implies the left hand side does also,

$$|\mathbb{1}\{\tilde{S}_t^4 = y+1\} - \mathbb{1}\{\tilde{S}_t^3 = y\}| \leq \mathbb{1}\{\tilde{S}_t^4 \neq \tilde{S}_t^3 - 1\}.$$

Then, additionally using the fact that $\tilde{S}^3$ and $\tilde{S}^4$ are independent of $\tilde{S}^1$,

$$\begin{aligned} |\tilde{p}_t(x, y+1) - \tilde{p}_t(x, y)| &\leq \mathbb{P}[\tilde{S}_t^1 = x]\mathbb{P}[\tilde{S}_t^4 \neq \tilde{S}_t^3 - 1] \\ &= \mathbb{P}[\tilde{S}_t^1 = x]\mathbb{P}[T > t]. \end{aligned}$$

We can use this inequality to bound the expression $|\tilde{p}_t(x, y+1) - \tilde{p}_t(x, y)|$ as follows,

- When t is small, the quantity $\mathbb{P}[\tilde{S}_t^1 = x]$ is small as the Poisson clock must ring at least $|x|$ times in the interval $[0, t]$ for the event $\{\tilde{S}_t^1 = x\}$ to have positive probability.
- When t is large, the quantity $\mathbb{P}[T > t]$ is small as it's likely the random walks on $\mathbb{Z}_k$ would align in such a way at least once in the interval $[0, t]$.

We bound these probabilities from above. For any $\theta > 0$, using the reasoning above in the first line followed by Markovs inequality in the third line,

$$\begin{aligned} \mathbb{P}[\tilde{S}_t^1 = x] &\leq \mathbb{P}[N^1(t) \geq |x|] \\ &= \mathbb{P}[e^{\theta N^1(t)} \geq e^{\theta |x|}] \\ &\leq \frac{\mathbb{E}[e^{\theta N^1(t)}]}{e^{\theta |x|}} \\ &= e^{-\theta |x|} \sum_{n=0}^{\infty} e^{\theta n} \cdot \mathbb{P}[N^1(t) = n] \\ &= e^{-\theta |x|} \sum_{n=0}^{\infty} e^{\theta n} \cdot e^{-\frac{1}{2}t} \frac{(\frac{1}{2}t)^n}{n!} \\ &= e^{-\theta |x| - \frac{1}{2}t} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}t \cdot e^{\theta})^n}{n!} \\ &= \exp\{-\theta |x| + \frac{1}{2}t(e^{\theta} - 1)\}. \end{aligned}$$

So in particular for $\theta = \log(2)$, it follows that for all $t \leq |x|$,

$$\mathbb{P}[\tilde{S}_t^1 = x] \leq \exp\{-|x|(\log(2) - \frac{1}{2})\} = \exp\{-|x| \cdot C_1\}$$

Similairly, we now bound $\mathbb{P}[T > t]$ from above. First of all, define the constant $C_2 = \mathbb{P}[T \leq 1]$,

$$C_2 > \mathbb{P}[\tilde{S}_1^3 = 1, \tilde{S}_1^2 = 0] > 0 \implies \mathbb{P}[T > 1] = 1 - C_2 < 1.$$

Then by the Markov property of the random walk,

$$\mathbb{P}[T > n] = (1 - C_2)^n \le e^{-nC_2}.$$

Thus,

$$\begin{aligned}
|D_2G(x,y)| &\le \int_0^\infty |\tilde{p}_t(x,y+1) - \tilde{p}_t(x,y)|\, dt \\
&\le \int_0^\infty \mathbb{P}[\tilde{S}_t^1 = x]\mathbb{P}[T > t]\, dt \\
&\le \int_0^{|x|} \mathbb{P}[\tilde{S}_t^1 = x]\, dt + \int_{|x|}^\infty \mathbb{P}[T > t]\, dt \\
&\le \int_0^{|x|} e^{-|x|C_1}\, dt + \int_{|x|}^\infty e^{-tC_2}\, dt \\
&= |x|e^{-|x|C_1} + C_2^{-1}e^{-|x|C_2}.
\end{aligned}$$

Finally, using Lemma 4.3 with $c = e^{-C_1}$,

$$\begin{aligned}
\sum_{(x,y)\in\mathbb{Z}\times\mathbb{Z}_k} |D_2G(x,y)| &= \sum_{y=1}^{k}\sum_{x\in\mathbb{Z}} |D_2G(x,y)| \\
&\le \sum_{y=1}^{k}\sum_{x\in\mathbb{Z}} \left[|x|\cdot e^{-|x|C_1} + C_2^{-1}e^{-|x|C_2}\right] \\
&= k\sum_{x\in\mathbb{Z}} \left[|x|\cdot e^{-|x|C_1} + C_2^{-1}e^{-|x|C_2}\right] \\
&< \infty
\end{aligned}$$

□

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